2,6 2 M. Math

(b) On the time scale $t \le C/\sqrt{\varepsilon}$ ($C > 0$ fixed),

$$|F(t,\varepsilon)| \le \varepsilon^2 \cdot \frac{C^2}{\varepsilon} = C^2\varepsilon,$$

so $F = \mathcal{O}(\varepsilon)$.

(c) To show $F \neq o(\varepsilon)$, consider $t_\varepsilon = 1/\sqrt{\varepsilon}$. Then

$$\frac{F(t_\varepsilon,\varepsilon)}{\varepsilon} = \cos\left(\varepsilon^{-3/2}\right).$$

Since $\cos(\varepsilon^{-3/2})$ oscillates between -1 and 1 and does not tend to 0 as $\varepsilon \to 0$, it follows that $F \neq o(\varepsilon)$ on this scale.

Solution of Exercise 3.

Consider the BVP

$$\varepsilon u''(x) - u(x) = -1, \quad u(0) = u(1) = 0,$$

with $\lambda = 1/\sqrt{\varepsilon}$.

1) **Verification of exact solution.** Let

$$u_\varepsilon(x) = 1 - Ae^{\lambda x} + Be^{-\lambda x}, \quad A = \frac{1-e^{-\lambda}}{e^{\lambda}-e^{-\lambda}}, \quad B = \frac{1-e^{\lambda}}{e^{\lambda}-e^{-\lambda}}.$$

Since $\varepsilon\lambda^2 = 1$, direct substitution shows $\varepsilon u_\varepsilon'' - u_\varepsilon = -1$. Boundary conditions : $u_\varepsilon(0) = 1-A+B = 0$ and $u_\varepsilon(1) = 1 - Ae^{\lambda} + Be^{-\lambda} = 0$, verified by algebraic simplification.

2) **Convergence in L^2.** As $\varepsilon \to 0$, $\lambda \to \infty$, and

$$A \sim e^{-\lambda}, \quad B \sim -1.$$

Hence,

$$u_\varepsilon(x) - 1 = -Ae^{\lambda x} + Be^{-\lambda x} \approx -e^{-\lambda(1-x)} - e^{-\lambda x}.$$

Both terms decay exponentially away from the boundaries. Compute :

$$\|u_\varepsilon - 1\|_{L^2}^2 \le 2\left(\int_0^1 e^{-2\lambda(1-x)}dx + \int_0^1 e^{-2\lambda x}dx\right) = \frac{2(1-e^{-2\lambda})}{\lambda} \to 0.$$

Thus $\|u_\varepsilon - 1\|_{L^2} \to 0$, so the problem is regularly perturbed in the L^2-norm.

3) **Formal expansion.** Assume $u = 1 + \varepsilon u_1 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3)$. Substituting into the ODE :

$$\varepsilon(u_1'' + \varepsilon u_2'' + \cdots) - (1 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots) = -1.$$

Matching powers :

$$\mathcal{O}(\varepsilon) : -u_1 = 0 \Rightarrow u_1 = 0, \qquad \mathcal{O}(\varepsilon^2) : -u_2 + u_1'' = 0 \Rightarrow u_2 = 0.$$

Hence $u_1(x) = 0$, $u_2(x) = 0$. **Deduction :** The regular expansion yields the constant solution $u = 1$, which satisfies the reduced equation but not the boundary conditions. However, the discrepancy is confined to thin boundary layers whose L^2-norm vanishes as $\varepsilon \to 0$.

Perturbations of Differential Equations

Solution of Final Exam

Solution of Exercise 1.

Consider the initial value problem

$$\frac{dy}{dt} + (1+\varepsilon)y = \varepsilon e^{-2t}, \quad y(0) = 1,$$

where $0 < \varepsilon \ll 1$.

1) **Exact solution.** Using the integrating factor $\mu(t) = e^{(1+\varepsilon)t}$, we obtain

$$\frac{d}{dt}\left(ye^{(1+\varepsilon)t}\right) = \varepsilon e^{(\varepsilon-1)t}.$$

Integrating from 0 to t and using $y(0) = 1$:

$$y(t)e^{(1+\varepsilon)t} = 1 + \varepsilon \int_0^t e^{(\varepsilon-1)s}\, ds = 1 + \frac{\varepsilon}{1-\varepsilon}(1 - e^{(\varepsilon-1)t}).$$

Hence,

$$y(t,\varepsilon) = e^{-(1+\varepsilon)t} + \frac{\varepsilon}{1-\varepsilon}\left(e^{-(1+\varepsilon)t} - e^{-2t}\right) = \frac{1}{1-\varepsilon}e^{-(1+\varepsilon)t} - \frac{\varepsilon}{1-\varepsilon}e^{-2t}.$$

2) **Taylor expansion of exact solution.** Expand up to $\mathcal{O}(\varepsilon^2)$. Using

$$e^{-(1+\varepsilon)t} = e^{-t}(1 - \varepsilon t + \mathcal{O}(\varepsilon^2)), \qquad \frac{1}{1-\varepsilon} = 1 + \varepsilon + \mathcal{O}(\varepsilon^2),$$

we find

$$y(t,\varepsilon) = e^{-t} + \varepsilon\left[e^{-t}(1-t) - e^{-2t}\right] + \mathcal{O}(\varepsilon^2).$$

Thus,

$$y_0(t) = e^{-t}, \qquad y_1(t) = e^{-t}(1-t) - e^{-2t}.$$

3) **Regular perturbation method.** Assume $y = y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2)$. Substituting into the ODE :

$$(y_0' + y_0) + \varepsilon(y_1' + y_1 + y_0) = \varepsilon e^{-2t} + \mathcal{O}(\varepsilon^2).$$

At $\mathcal{O}(1)$: $y_0' + y_0 = 0$, $y_0(0) = 1 \Rightarrow y_0(t) = e^{-t}$.
At $\mathcal{O}(\varepsilon)$: $y_1' + y_1 = e^{-2t} - e^{-t}$, $y_1(0) = 0$. Solving with integrating factor e^t :

$$\frac{d}{dt}(y_1 e^t) = e^{-t} - 1 \;\Rightarrow\; y_1(t) = e^{-t}(1-t) - e^{-2t}.$$

4) **Verification.** The expressions for y_0 and y_1 obtained in parts (2) and (3) are identical. Therefore, the regular perturbation expansion coincides with the Taylor expansion in ε of the exact solution up to $\mathcal{O}(\varepsilon^2)$.

Solution of Exercise 2.

Let $F(t,\varepsilon) = \varepsilon^2 t^2 \cos\left(\frac{t}{\varepsilon}\right)$.

(a) On the time scale $t \in [0,T]$ (fixed $T > 0$), we have

$$|F(t,\varepsilon)| \le \varepsilon^2 T^2.$$

Thus $F = \mathcal{O}(\varepsilon^2)$ with explicit constants $K = T^2$, $\varepsilon_0 = 1$, and $\delta = T$.