I مج 1M Math

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Commutative Algebra
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Final examination - Master 1

Full Name: .... Group: ....

EXERCISE 0.1 (6pts). Let R be a commutative ring. Prove the following

(1) If $\mathfrak{p}$ is a prime ideal, then $\sqrt{\mathfrak{p}} = \mathfrak{p}$.
(2) If R is a PID. Consider the sum $\langle a\rangle + \langle b\rangle = \langle c\rangle$. Show that $c = gcd(a, b)$.
(3) Give an example of a prime ideal which is not maximal (Think of the ring of multivariable polynomials).

(1) We have $p \subset \sqrt{p}$, let $x \in \sqrt{p}$; $\exists m > 0$ s.t $x^m \in p$. since p is prime, $x \in p$ or $x^{m-1} \in p$; after a f. n. of steps we get $x \in p$.
(2) let $x | a$ and $x | b$ since $c = a\alpha + b\beta$; we deduce that $x | c$. Now it is clear that $c | a$ and $c | b$. Hence $c = gcd(a, b)$.

(3) Let $R = k[X, Y]$; $I = \langle Y\rangle$; ~~so~~ We see that $R/I = k[X,Y]/\langle Y\rangle = k[X]$ is integral but not a field; hence $\langle Y\rangle$ is prime but not maximal.

EXERCISE 0.2 (3pts). Let R be a commutative ring and $S = R \setminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal.

(a) Show that S is a multiplicative set.
(b) Show that a ring has exactly one maximal ideal if and only if the nonunits form an ideal (This called local ring).
(c) Let $S^{-1}R$ be the localization R with respect to S. Show that this ring is local.

(a) $1 \in S$ clear; $x, y \in S \Leftrightarrow x, y \notin p$ hence $x \cdot y \notin p$, so $x \cdot y \in S$.
(b) let $U = \{\text{non-units}\}$; if R has one maximal ideal m then $\langle U\rangle \subseteq m$; but $m \subseteq U$; hence $m = U$. Conversly; if U is an ideal; it is the only one because any ideal is contained in U.

(3) the set of non units of $S^{-1}R$ is exactly $S^{-1}p$ which is an ideal; by (2) we get the result.

EXERCISE 0.3 (6 pts). Let R be a commutative ring and I, J two ideals of R. .

(1) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

(2) Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.

(3) Prove that R/I is reduced (i.e. has no non-zero nilpotents) if and only if $I = \sqrt{I}$.

(1) $I \cap J \subset \sqrt{I}$ so $\sqrt{I \cap J} \subset \sqrt{\sqrt{I}} = \sqrt{I}$; also $\sqrt{I \cap J} \subset \sqrt{J}$ hence $\sqrt{I \cap J} \subset \sqrt{I} \cap \sqrt{J}$. Let $x \in \sqrt{I} \cap \sqrt{J}$; $\exists n, m$ s.t $x^n \in I$, $x^m \in J$

So $x^{n+m} \in I \cap J \Rightarrow x \in \sqrt{I \cap J}$

(2) let $x \in \sqrt{\sqrt{I}}$; $\exists n > 0$ $x^n \in \sqrt{I}$ so $\exists m > 0$ $(x^n)^m = x^{nm} \in I$ hence $x \in \sqrt{I}$, The converse is clear.

(3) ~~let~~ Assume R/I reduced; let $x \in \sqrt{I}$, so $x^n \in I$ hence

$\overline{x^n} = 0$, but $\overline{x^n} = \overline{x}^n$, so $\overline{x}^n = 0$, hence $\overline{x}$ is nilpote[nt]

hence $\overline{x} = 0 \Rightarrow x \in I \Rightarrow I = \sqrt{I}$.

Conversely: if $I = \sqrt{I}$; let $\overline{x} \in R/I$ be nilpotent

So $\overline{x}^n = 0 \Rightarrow \overline{x^n} = 0 \Rightarrow x^n \in I \Rightarrow x \in \sqrt{I} = I \Rightarrow x \in I$

$\Rightarrow \overline{x} = 0$.

EXERCISE 0.4 (6pts). Let R be a Noetherian ring, $M = (m)$ a cyclic R-module and $N \subset M$ a submodule.

(1) Define $I := \{ r \in R \mid rm \in N \}$. Show that I is an ideal of R.

(2) Prove that $N = Im$.

(3) Using the fact that R is Noetherian, prove that N is finitely generated.

(1) $0 \in I$; if $x, y \in I$, $xm \in N$ and $ym \in N \Rightarrow (x-y)m \in N$

$\Rightarrow x - y \in I$. if $a \in I$, $x \in R$, $\Rightarrow am \in N \Rightarrow xam \in N \Rightarrow xa \in I$.

(2) $Im \subset N$ by definition. Let $x \in N$; $\exists r \in R$; s.t

$x = rm \Rightarrow r \in I$ (by def. of I) hence $N \subset Im$.

(3) let $r_1, \dots, r_n$ be a set that generate I; so if $x \in N$

$\exists r \in I$ s.t $x = rm$;

$\exists \alpha_1, \dots, \alpha_n$ s.t $r = \alpha_1 r_1 + \dots + \alpha_n r_n$; hence

$x = \alpha_1 r_1 m + \dots + \alpha_n r_n m$; So $N = \langle r_1 m, \dots, r_n m \rangle$